STRONG CONVERGENCE OF ITERATIVE METHODS FOR CONTINUOUS PSEUDOCONTRACTIVE MAPPINGS (Nonlinear Analysis and Convex Analysis)

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ABSTRACT. In this paper, we consider an iterative method for a continuous pseudocontractive mapping T and a continuous bounded strongly pseudocontractive mapping A in a reflexive Banach space having a uniformly Gâteaux differentiable norm. Under suitable conditions on control parameters, we establish strong convergence of the sequence generated by the proposed iterative algorithm to a fixed point of the mapping T, which solves a ceratin variational inequality related to A.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we denote by E with the norm $\|\cdot\|$ and E^* a real Banach space and the dual space of E, respectively . Let C be a nonempty closed convex subset of E. For the mapping $T : C \to C$, we denote the fixed point set of T by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$.

Let J denote the normalized duality mapping from E into 2^{X^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|,\ \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pair between E and E^*. Recall that the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{1.1}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is called a *smooth* Banach space. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (1.1) is attained uniformly for $(x, y) \in U \times U$. It is known that E is smooth if and only if the normalized duality mapping J is single-valued. It is well known that if E is uniformly smooth, then the duality mapping is norm-to-norm uniformly continuous on bounded subsets of E, and that if E has a uniformly Gâteaux differentiable norm, J is norm-to-weak* uniformly continuous on each bounded subsets of E ([1, 2]).

It is relevant to the our results of this paper to note that while every uniformly smooth Banach space is a reflexive Banach space having a uniformly Gâteaux differentiable norm, the converse does not hold. To see this, consider E to be the direct sum $l^2(l^{p_n})$, the class of all those sequences $x = \{x_n\}$ with $x_n \in l^{p_n}$ and $\|x\| = (\sum_{n<\infty}\|x_n\|^2)^{\frac{1}{2}}$ (see [3]). If $1 < p_n < \infty$ for $n \in \mathbb{N}$, where either $\limsup_{n\to\infty} p_n = \infty$ or $\liminf_{n\to\infty} p_n = 1$, then E is a reflexive Banach space with a uniformly Gâteaux differentiable norm, but is not uniformly smooth (see [3, 4, 5]). We also observe that the spaces which enjoy the fixed point property

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(shortly, F.P.P) for nonexpansive mappings are not necessarily spaces having a uniformly Gâteaux differentiable norm. On the other hand, the converse of this fact appears to be unknown as well.

A Banach space E is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, \quad x \neq y \ \text{ implies } \ \frac{\|x+y\|}{2} < 1.$$

A Banach space E is said to be *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all $\varepsilon > 0$, where $\delta_E(\varepsilon)$ is the *modulus of convexity* of E defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \ \|y\| \leq 1, \ \|x-y\| \geq \varepsilon\right\}, \ \varepsilon \in [0,2].$$

It is well known that a uniformly convex Banach space E is reflexive and strictly convex ([1]) and satisfies the F.P.P. for nonexpansive mappings. However, it appears to be unknown whether a reflexive and strictly convex space satisfies the F.P.P. for nonexpansive mappings.

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in E is called *pseudocontractive* if the inequality

$$\|x-y\| \leq \|x-y+r((I-T)x-(I-T)y)\| \tag{1.2}$$

holds for each , $y \in D(T)$ and for all $r > 0$. From a result of Kato [6],we know that (1.1) is equivalent to (1.3) below; there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx-Ty, j(x-y)\rangle \leq \|x-y\|^2 \tag{1.3}$$

for all $x, \ y \in D(T)$. The mapping T is said to be *strongly pseudocontractive* it there exists a constant $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx-Ty, j(x-y)\rangle \leq k\|x-y\|^2$$

for all $x, \ y \in D(T)$

The class of pseudocontractive mappings is one of the most important classes of mappings in nonlinear analysis and it has been attracting mathematician's interest. In addition to generalizing the nonexpansive mappings (the mappings $T : D \to E$ for which $\|Tx-Ty\| \leq \|x-y\|, \ \forall x, \ y \in D$), the pseudocontractive ones are characterized by the fact that T is pseudocontractive if and only if $I-T$ is accretive, where a mapping A with domain $D(A)$ and range $R(A)$ in E is called accretive if the inequality

$$\|x-y\| \leq \|x-y+s(Ax-Ay)\|,$$

holds for every $x, \ y \in D(A)$ and for all $s > 0$.

Within the past 40 years or so, many authors have been devoting their study to the existence of zeros of accretive mappings or fixed points of pseudocontractive mappings and iterative construction of zeros of accretive mappings and of fixed points of pseudocontractive mappings (see [5, 7, 8, 9, 10]). Also several iterative methods for approximating fixed points (zeros) of nonexpansive and pseudocontractive mappings (accretive mappings) in Hilbert spaces and Banach spaces have been introduced and studied by many authors. We can refer to [11, 12, 13, 14, 15, 16, 17] and references in therein.

In 2007, Yao et al. [15] introduced an iterative method (1.4) below for approximating fixed points of a continuous pseudocontractive mapping T without compactness assumption on its domain in a uniformly smooth Banach space: for arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$,

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n, \quad \forall n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ satisfying some appropriate conditions. By using the Reich inequality ([9]) in uniformly smooth Banach spaces:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|), \ \ \forall x, y \in E, \tag{1.5}$$

where $b : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function, they proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T. In particular, in 2007, by using the viscosity iterative method studied by [18, 19], Song and Chen [16] introduced a modified implicit iterative method (1.6) below for a continuous pseudocontractive mapping T without compactness assumption on its domain in a real reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm: for arbitrary initial value $x_0 \in C$,

$$\begin{cases} x_n = \alpha_n y_n + (1-\alpha_n)Tx_n, \\ y_n = \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying some appropriate conditions and $f : C \to C$ is a contractive mapping, and proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a fixed point of T, which is the unique solution of a ceratin variational inequality related to f.

In this paper, inspired and motivated by above-mentioned results, we introduce the following iterative method for a continuous pseudocontractive mapping T: for arbitrary initial value $x_0 \in C$,

$$x_n = \alpha_n Ax_n + \beta_n x_{n-1} + (1-\alpha_n-\beta_n)Tx_n, \quad \forall n \geq 1, \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ and $A : C \to C$ is a bounded continuous strongly pseudocontractive mapping with a pseudocontractive constant $k \in (0,1)$. In either a reflexive Banach space having a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings, or a reflexive and strict convex Banach space having a uniformly Gâteaux differentiable norm, we establish the strong convergence of the sequence generated by proposed iterative method (1.7) to a fixed point of the mapping, which solves a ceratin variational inequality related to A. The main result generalizes, improves and develops the corresponding results of Yao et al. [15] and Song and Chen [16] as well as Rafiq [17].

We need the following well-known lemmas for the proof of our main result.

Lemma 1.1 ([1, 2]). *Let E be a Banach space and J be the normalized duality mapping on E. Then for any $x,\ y \in E$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 1.2 ([20]). *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$,

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=0}^{\infty}\lambda_n|\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

2. Iterative methods

We need the following result which was given in [10].

Proposition 2.1. *Let C be a closed convex subset of a Banach space E. Suppose that T, A are two continuous mappings from C into itself, which are pseudocontractive and strongly pseudocontractive, respectively. Then there exists a unique path $t \mapsto x_t \in C$, $t \in (0,1)$, satisfying*

$$x_t = tAx_t + (1-t)Tx_t.$$

Further, the followings hold:

(i) *Suppose that there exists a bounded sequence* $\{x_n\}$ *in* C *such that* $x_n - Tx_n \to 0$, *while* $\{x_n - Ax_n\}$ *is bounded. Then the path* $\{x_t\}$ *is bounded.*
(ii) *In particular, if* T *has a fixed point in* C, *then the path* $\{x_t\}$ *is bounded.*
(iii) *If* p *is a fixed point of* T, *there exists* $j \in J(x_t - p)$ *such that*

$$\langle x_t - Ax_t, j\rangle \le 0.$$

We prepare the following result for the existence of a solution of the variational inequality related to A. For the proof, see [10, 22].

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a Banach space* E *and* T *be a continuous pseudocontractive mapping from* C *into itself with* $F(T) \neq \emptyset$ *and* $A : C \to C$ *be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient* $k \in (0,1)$. *For each* $t \in (0,1)$, *let* $x_t \in C$ *be defined by*

$$x_t = tAx_t + (1-t)Tx_t. \tag{2.1}$$

If one of the following assumptions holds:

(H1) *E is a reflexive Banach space, the norm of E is uniformly Gâteaux differentiable, and every weakly compact convex subset of E has the fixed point property for nonexpansive mappings;*
(H2) *E is a reflexive and strictly convex Banach space and the norm of E is uniformly Gâteaux differentiable,*

then the path $\{x_t\}$ *converges strongly to a point* u *in* $F(T)$, *which is the unique solution of the variational inequality*

$$\langle (I-A)u, J(u-v)\rangle \le 0, \quad \forall v \in F(T). \tag{2.2}$$

Using Theorem 2.1, we establish our main result.

Theorem 2.2. *Let* E *be a Banach space and* C *be a nonempty closed convex subset of* E. *Let* $T : C \to C$ *be a continuous pseudocontractive mapping such that* $F(T) \neq \emptyset$, *and* $A : C \to C$ *be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive constant* $k \in (0,1)$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0,1)$ *satisfying the following conditions:*
(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \beta_n = 0$;
(C2) $\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_n+\beta_n} = \infty$.
For arbitrary initial value $x_0 \in C$, *let the sequence* $\{x_n\}$ *be defined by*

$$x_n = \alpha_n Ax_n + \beta_n x_{n-1} + (1-\alpha_n-\beta_n)Tx_n, \quad \forall n \ge 1. \tag{2.6}$$

If one of the following assumptions holds:

(H1) *E is a reflexive Banach space, the norm of E is uniformly Gâteaux differentiable, and every weakly compact convex subset of E has the fixed point property for nonexpansive mappings;*
(H2) *E is a reflexive and strictly convex Banach space and the norm of E is uniformly Gâteaux differentiable,*

then $\{x_n\}$ *converges strongly to a fixed point* p *of* T, *which is the unique solution of the variational inequality*

$$\langle (I-A)p, J(p-q)\rangle \le 0, \quad \forall q \in F(T). \tag{2.7}$$

Proof. We divide the proof into several steps as follows.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $q \in F(T)$. Then, noting that

$$x_n - q = \alpha_n(Ax_n - q) + \beta_n(x_{n-1} - q) + (1 - \alpha_n - \beta_n)(Tx_n - q),$$

$$\langle Tx_n - q, J(x_n - q)\rangle \le \|x_n - q\|^2 \tag{2.8}$$

and

$$\langle Ax_n - Aq, J(x_n - q)\rangle \le k\|x_n - q\|^2, \tag{2.9}$$

we have

$$\begin{aligned}\|x_n - q\|^2 &= \langle \alpha_n[(Ax_n - Aq) + (Aq - q)] + \beta_n(x_{n-1} - q) \\ &\qquad + (1 - \alpha_n - \beta_n)(Tx_n - q), J(x_n - q)\rangle \\ &\le \alpha_n k\|x_n - q\|^2 + \alpha_n\|Aq - q\|\|x_n - q\| \\ &\qquad + \beta_n\|x_{n-1} - q\|\|x_n - q\| + (1 - \alpha_n - \beta_n)\|x_n - q\|^2,\end{aligned}$$

which implies

$$\begin{aligned}\|x_n - q\| \le (1 - \alpha_n(1 - k) - \beta_n)\|x_n - q\| + \alpha_n\|Aq - q\| \\ + \beta_n\|x_{n-1} - q\|.\end{aligned}$$

So, we obtain

$$\begin{aligned}\|x_n - q\| &\le \frac{\alpha_n}{(1-k)\alpha_n + \beta_n}\|Aq - q\| + \frac{\beta_n}{(1-k)\alpha_n + \beta_n}(x_{n-1} - q)\| \\ &= \frac{(1-k)\alpha_n}{(1-k)\alpha_n + \beta_n}\frac{\|Aq - q\|}{1-k} + \frac{\beta_n}{(1-k)\alpha_n + \beta_n}\|x_{n-1} - p\| \\ &\le \max\left\{\|x_{n-1} - q\|, \frac{\|Aq - q\|}{1-k}\right\}.\end{aligned}$$

By induction, we have

$$\|x_n - q\| \le \max\left\{\|x_0 - q\|, \frac{1}{1-k}\|Aq - q\|\right\} \quad \text{for } n \ge 1.$$

Hence $\{x_n\}$ is bounded. Since A is a bounded mapping, $\{Ax_n\}$ is bounded. From (2.6), it follows that

$$\|Tx_n\| = \frac{1}{1 - \alpha_n - \beta_n}(\|x_n\| + \alpha_n\|Ax_n\| + \beta_n\|x_{n-1}\|),$$

and so $\{Tx_n\}$ is bounded (as $n \to \infty$).

Step 2. We show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. In fact, by (2.1) and the condition (C1), we have

$$\|x_n - Tx_n\| \le \alpha_n\|Ax_n - Tx_n\| + \beta_n\|x_{n-1} - Tx_n\| \to 0.$$

Step 3. We show that

$$\limsup_{n\to\infty}\langle Ap - p, J(x_n - p)\rangle \le 0,$$

where $p = \lim_{t\to 0} x_t$ with $x_t \in C$ being defined by $x_t = tAx_t + (1-t)Tx_t$. To this end, we note that

$$\begin{aligned}x_t - x_n &= tAx_t + (1-t)Tx_t - x_n \\ &= t(Ax_t - x_t) + (Tx_t - x_n) - t(Tx_t - x_t) \\ &= t(Ax_t - x_t) + (Tx_t - Tx_n) + (Tx_n - x_n) + t^2(Ax_t - Tx_t).\end{aligned}$$

Then, it follows that

$$\begin{aligned}\|x_t - x_n\|^2 &= t\langle Ax_t - x_t, J(x_t - x_n)\rangle + \langle Tx_t - Tx_n, J(x_t - x_n)\rangle \\ &\qquad + \langle Tx_n - x_n, J(x_t - x_n)\rangle + t^2\langle Ax_t - Tx_t, J(x_t - x_n)\rangle \\ &\le t\langle Ax_t - x_t, J(x_t - x_n)\rangle + \|x_t - x_n\|^2 \\ &\qquad + \|Tx_n - x_n\|\|x_t - x_n\| + t^2\|Ax_t - Tx_t\|\|x_t - x_n\|,\end{aligned}$$

which implies that

$$\langle Ax_t - x_t, J(x_n - x_t)\rangle \le \frac{\|Tx_n - x_n\|}{t}\|x_t - x_n\| + t\|Ax_t - Tx_t\|\|x_t - x_n\|. \tag{2.10}$$

From Proposition 2.1, we know that $\{x_t\}$, $\{Ax_t\}$ and $\{Tx_t\}$ are bounded. Since $\{x_n\}$ and $\{Tx_n\}$ are also bounded and $x_n - Tx_n \to 0$ by Step 2, taking the upper limit as $n \to \infty$ in (2.10), we get

$$\limsup_{n\to\infty}\langle Ax_t - x_t, J(x_n - x_t)\rangle \le tL, \tag{2.11}$$

where $L > 0$ is a constant such that $\|Ax_t - Tx_t\|\|x_t - x_n\| \le L$ for all $n \ge 0$ and $t \in (0,1)$. Taking the lim sup as $t \to 0$ in (2.11) and noticing the fact that the two limits are interchangeable due to the fact that J is norm-to-weak* uniformly continuous on each bounded subsets of E, we have

$$\limsup_{n\to\infty}\langle Ap - p, J(x_n - p)\rangle \le 0.$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - p\| = 0$, where $p = \lim_{t\to 0} x_t$ with $x_t \in C$ being defined by $x_t = tAx_t + (1-t)Tx_t$ and p is the unique solution of the variational inequality (2.7) by Theorem 2.1. First, from (2.6), (2.8) and (2.9), we have

$$\begin{aligned}
\|x_n - p\|^2 &= \langle x_n - p, J(x_n - p)\rangle \\
&= \langle \alpha_n(Ax_n - p) + \beta_n(x_{n-1} - p) + (1 - \alpha_n - \beta_n)(Tx_n - p), J(x_n - p)\rangle \\
&= \langle \alpha_n(Ax_n - Ap), J(x_n - p)\rangle + \beta_n\langle x_{n-1} - p, J(x_n - p)\rangle \\
&\quad + (1 - \alpha_n - \beta_n)\langle Tx_n - p, J(x_n - p)\rangle + \alpha_n\langle Ap - p, J(x_n - p)\rangle \\
&\le \alpha_n k\|x_n - p\|^2 + \beta_n\|x_{n-1} - p\|\|x_n - p\| \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - p\|^2 + \alpha_n\langle Ap - p, J(x_n - p)\rangle \\
&\le \alpha_n k\|x_n - p\|^2 + \frac{\beta_n}{2}(\|x_{n-1} - p\|^2 + \|x_n - p\|^2) \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - p\|^2 + \alpha_n\langle Ap - p, J(x_n - p)\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_n - p\|^2 &\le \frac{\beta_n}{2(1-k)\alpha_n + \beta_n}\|x_{n-1} - p\|^2 \\
&\quad + \frac{2\alpha_n}{2(1-k)\alpha_n + \beta_n}\langle Ap - p, J(x_n - p)\rangle \\
&= \left(1 - \frac{2(1-k)\alpha_n}{2(1-k)\alpha_n + \beta_n}\right)\|x_{n-1} - p\|^2 \\
&\quad + \frac{2(1-k)\alpha_n}{2(1-k)\alpha_n + \beta_n}\frac{\langle Ap - p, J(x_n - p)\rangle}{1-k} \\
&= (1 - \lambda_n)\|x_{n-1} - p\|^2 + \lambda_n\delta_n,
\end{aligned} \tag{2.12}$$

where $\lambda_n = \frac{2(1-k)\alpha_n}{2(1-k)\alpha_n+\beta_n}$ and $\delta_n = \frac{1}{1-k}\langle Ap-p, J(x_n-p)\rangle$. We observe that $0 \le \frac{2(1-k)\alpha_n}{2(1-k)\alpha_n+\beta_n} \le 1$ and $\frac{(1-k)\alpha_n}{\alpha_n+\beta_n} = \frac{2(1-k)\alpha_n}{2\alpha_n+2\beta_n} < \frac{2(1-k)\alpha_n}{2(1-k)\alpha_n+\beta_n}$. From the condition (C2) and Step 3, it is easily seen that $\sum_{n=1}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Thus, applying Lemma 1.2 to (2.12), we conclude that $\lim_{n\to\infty} x_n = p$. This completes the proof. □

Corollary 2.1. *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping such that $F(T) \neq \emptyset$ and $A : C \to C$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive constant $k \in (0,1)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ satisfying the conditions* (C1) *and* (C2) *in Theorem 2.2. For arbitrary initial value $x_0 \in C$,* *let the sequence $\{x_n\}$ be generated by* (2.6) *in Theorem 2.2. Then $\{x_n\}$ converges strongly to a fixed point p of T, which is the unique solution of the variational inequality* (2.7)

Corollary 2.2 ([16, Theorem 3.1]). *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in* $(0, 1)$ *satisfying the conditions* (C1) *and* (C2) *in Theorem 2.2 and $\gamma_n = 1 - \alpha_n - \beta_n$ for $n \geq 1$. For arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$, let the sequence $\{x_n\}$ be generated by*

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point p of T, which is the unique solution of the variational inequality

$$\langle p - u, J(p - q)\rangle \leq 0, \quad \forall q \in F(T).$$

Proof. Taking $Ax = u$, $\forall x \in C$ as a constant function, the result follows from Corollary 2.1.

Corollary 2.3. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping such that $F(T) \neq \emptyset$ and $A : C \to C$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive constant $k \in (0, 1)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be three sequences in* $(0, 1)$ *satisfying the conditions* (C1) *and* (C2) *in Theorem 2.2. For arbitrary initial value $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (2.6) *in Theorem 2.2. Then $\{x_n\}$ converges strongly to a fixed point p of T, which is the unique solution of the variational inequality* (2.7).

Remark 2.1.

1) Theorem 2.2 extends and improves Theorem 3.1 of Yao et al. [15] in the following aspects:
   (a) u is replaced by a continuous bounded strongly pseudocontractive mapping A.
   (b) The uniformly smooth Banach space is extended to a reflexive Banach space having a uniformly Gâteaux differentiable norm.
   (c) The condition $\frac{\alpha_n}{\beta_n} \to 0$ in [15] is weakened to $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$.
2) It is worth pointing out that in Corollary 2.1 and Corollary 2.2, we do not use the Reich inequality (1.5) in comparison with Theorem 3.1 of Yao et al. [15].
3) Theorem 2.2 and Corollary 2.3 also develop and complement Theorem 3.1 and Corollary 3.2 of Song and Chen [16] by replacing the contractive mapping with a continuous bounded strongly pseudocontractive mapping in the iterative scheme (1.7) in [16].
4) The assumption (H1) in Theorem 2.1 and Theorem 2.2 appears to be independent of the assumption (H2).
5) We point out that the results in this paper apply to all L^p spaces, $1 < p < \infty$.

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